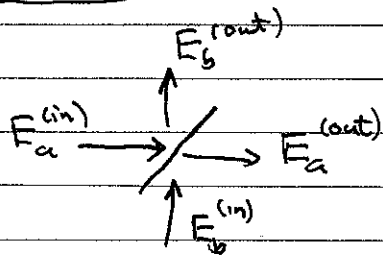


Physics 566

Problem Set #7: Solutions

The beam splitter and other linear transformations



$$\begin{bmatrix} E_a^{(out)} \\ E_b^{(out)} \end{bmatrix} = \begin{bmatrix} t & r \\ r & t \end{bmatrix} \begin{bmatrix} E_a^{(in)} \\ E_b^{(in)} \end{bmatrix}$$

↑
"S-matrix" S

(a) Unitarity of the S-matrix: $S^\dagger S = \mathbb{1}$

$$S S^\dagger = \begin{bmatrix} t & r \\ r & t \end{bmatrix} \begin{bmatrix} t^* & r^* \\ r^* & t^* \end{bmatrix} = \begin{bmatrix} |t|^2 + |r|^2 & tr^* + t^*r \\ t^*r + tr^* & |t|^2 + |r|^2 \end{bmatrix}$$

$$\Rightarrow \boxed{|t|^2 + |r|^2 = 1}$$

$$\text{Re}(tr^*) = 0$$

$$\Rightarrow \boxed{\text{Arg}(t) - \text{Arg}(r) = \pm \pi/2}$$

$$\text{let } T = |t|^2$$

$$\Rightarrow t = \sqrt{T} e^{i\phi_t} \quad r = i\sqrt{1-T} e^{i\phi_t}$$

ϕ_t depends on details on beam splitter

for $\phi_t = 0$

$$\begin{aligned} E_a^{(out)} &= \sqrt{T} E_a^{(in)} + i\sqrt{1-T} E_b^{(in)} \\ E_b^{(out)} &= \sqrt{T} E_b^{(in)} + i\sqrt{1-T} E_a^{(in)} \end{aligned}$$

(b) Quantized mode: $E_a \Rightarrow \hat{a}$ $E_b \Rightarrow \hat{b}$

Suppose no field is injected into port "b"

Classically $E_a^{(out)} = \sqrt{T} E_a^{(in)}$

Quantum analogy $\hat{a}^{(out)} = \sqrt{T} \hat{a}^{(in)}$?

No $[\hat{a}^{(out)}, \hat{a}^{(out)\dagger}] = T [\hat{a}^{(in)}, \hat{a}^{(in)\dagger}] = T \leq 1$

(c) ~~So~~ the uncertainty principle would be violated.

The problem is that we allowed attenuation of vacuum fluctuations. Formally, we violated unitarity in the transformation between input and output:

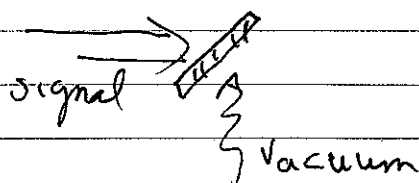
$$\hat{a}^{(out)} = \hat{S}^\dagger \hat{a}^{(in)} \hat{S} = \underbrace{\sqrt{T}}_t \hat{a}^{(in)} + i \underbrace{\sqrt{1-T}}_r \hat{b}^{(in)}$$

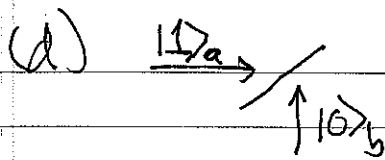
$$\Rightarrow [\hat{a}^{(out)}, \hat{a}^{(out)\dagger}] = |t|^2 [\hat{a}^{(in)}, \hat{a}^{(in)\dagger}] + |r|^2 [\hat{b}^{(in)}, \hat{b}^{(in)\dagger}]$$

$$+ t r^* [\hat{a}^{(in)}, \hat{b}^{(in)\dagger}] + t^* r [\hat{a}^{(in)\dagger}, \hat{b}^{(in)}]$$

$$\Rightarrow [\hat{a}^{(out)}, \hat{a}^{(out)\dagger}] = |t|^2 + |r|^2 = 1$$

One way to interpret this is that although we do not input a signal into port-b, vacuum fluctuations always enter that port





Input single photon into mode - a and nothing in mode - b

$$\Rightarrow |\Psi_{in}\rangle = |1\rangle_a \otimes |0\rangle_b = \hat{a}^{(in)\dagger} |0,0\rangle \leftarrow \text{total vacuum}$$

$$|\Psi_{out}\rangle = \hat{S} |\Psi_{in}\rangle = \hat{a}^{(out)\dagger} |0,0\rangle$$

$$= (t \hat{a}^{(in)\dagger} + r \hat{b}^{(in)\dagger}) |0,0\rangle$$

$$= t |1,0\rangle + r |0,1\rangle$$

$$|\Psi_{out}\rangle = t |1\rangle_a \otimes |0\rangle_b + r |0\rangle_a \otimes |1\rangle_b$$

(e) Now suppose we inject a coherent state



$$|\Psi_{in}\rangle = |\alpha\rangle_a \otimes |0\rangle_b$$

$$= \hat{D}_a^{(in)}(\alpha) |0,0\rangle$$

(displacement operator)

$$\hat{D}_a^{(in)}(\alpha) \equiv \exp\{\alpha \hat{a}^{(in)} + \alpha^* \hat{a}^{(in)\dagger}\}$$

$$\Rightarrow |\Psi_{out}\rangle = \hat{S} |\Psi_{in}\rangle = \hat{S} \hat{D}_a^{(in)} \hat{S}^\dagger \hat{S} |0,0\rangle$$

$$= \exp\{\alpha \hat{S} \hat{a}^{(in)} \hat{S}^\dagger + \alpha^* \hat{S} \hat{a}^{(in)\dagger} \hat{S}^\dagger\} |0,0\rangle$$

$$= \exp\{\alpha \hat{a}^{(out)} + \alpha^* \hat{a}^{(out)\dagger}\} |0,0\rangle$$

$$= \exp\{\alpha (t \hat{a}^{(in)} + r \hat{b}^{(in)}) + \alpha^* (t^* \hat{a}^{(in)\dagger} + r^* \hat{b}^{(in)\dagger})\}$$

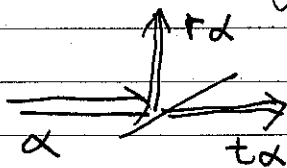
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Since $\hat{a}^{(in)}$ and $\hat{b}^{(in)}$ modes commute

$$|\Psi_{out}\rangle = \hat{D}_a^{(in)}(t\alpha) \hat{D}_b^{(in)}(r\alpha) |0, 0\rangle$$

$$\Rightarrow |\Psi_{out}\rangle = |t\alpha\rangle_a \otimes |r\alpha\rangle_b$$

This is the classically expected result



This is in contrast to the state

$$t|t\alpha\rangle_a \otimes |0\rangle_b + r|0\rangle_a \otimes |r\alpha\rangle_b$$

which is a "Schrodinger cat" state. This state describes a superposition of two macroscopic outcomes: the entire beam is transmitted (with probability $|t|^2$) or the entire beam is reflected (with probability $|r|^2$). This is a very nonclassical transformation, not accomplished by the linear beam splitter. The coherent state is basically a many photon copy of the single photon state. Each photon acts independently and randomly takes the transmitted or reflected path. The Poisson statistics are preserved.

(g) Linear optics \Rightarrow linear transformation of modes

$$E_k^{(out)} = \sum_{k'} U_{kk'} E_{k'}^{(in)}$$

\Uparrow
unitary matrix

Quantum mechanically, $\hat{a}_k^{(out)} = \hat{S} \hat{a}_k^{(in)} \hat{S}^\dagger = \sum_{k'} U_{kk'} \hat{a}_{k'}^{(in)}$

Suppose we start in an arbitrary multimode coherent state

$$\begin{aligned} |\psi_{in}\rangle &= \hat{D}^{(in)}(\{\alpha_k\}) |0\rangle = \prod_k \hat{D}^{(in)}(\alpha_k) |0\rangle \\ &= \prod_k \exp(\alpha_k \hat{a}_k^{(in)\dagger} - \alpha_k^* \hat{a}_k^{(in)}) = \exp\left[\sum_k (\alpha_k \hat{a}_k^{(in)\dagger} - \alpha_k^* \hat{a}_k^{(in)})\right] \end{aligned}$$

$$\begin{aligned} |\psi_{out}\rangle &= \hat{S} |\psi_{in}\rangle = \exp\left[\sum_k (\alpha_k \hat{a}_k^{(out)\dagger} - \alpha_k^* \hat{a}_k^{(out)})\right] |0\rangle \\ &= \exp\left[\sum_{k,k'} (\alpha_k U_{kk'}^* \hat{a}_{k'}^{(in)\dagger} - \alpha_k^* U_{kk'} \hat{a}_{k'}^{(in)})\right] |0\rangle \\ &= \exp\left[\sum_{k'} \left\{ \left(\sum_k U_{k'k} \alpha_k\right) \hat{a}_{k'}^{(in)\dagger} - \left(\sum_k U_{k'k} \alpha_k^*\right) \hat{a}_{k'}^{(in)} \right\}\right] |0\rangle \end{aligned}$$

where I have used $U_{kk'}^* = U_{k'k}$ for unitary matrix

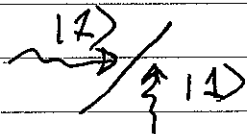
$$\Rightarrow |\psi_{out}\rangle = \exp\left[\sum_{k'} (\tilde{\alpha}_{k'}^{(out)} \hat{a}_{k'}^{(in)\dagger} - \tilde{\alpha}_{k'}^{(out)*} \hat{a}_{k'}^{(in)})\right] |0\rangle$$

where $\tilde{\alpha}_{k'}^{(out)} = \sum_k U_{k'k} \alpha_k$

$$\Rightarrow |\psi_{out}\rangle = \hat{D}(\{\tilde{\alpha}_k\}) |0\rangle$$

(h) A non-classical input leads to nonclassical phenomena, even for linear transformations

Suppose $|2\rangle^{(in)} = |1\rangle_a \otimes |1\rangle_b$, two "mode matched" single photons incident simultaneously on a beam splitter:



$$\Rightarrow |2\rangle^{(in)} = \hat{a}_{in}^{\dagger} \hat{b}_{in}^{\dagger} |0\rangle$$

$$\Rightarrow |2\rangle^{(out)} = \hat{a}_{out}^{\dagger} \hat{b}_{out}^{\dagger} |0\rangle = \frac{1}{\sqrt{2}} (\hat{a}^{\dagger} - i\hat{b}^{\dagger}) (\hat{b}^{\dagger} - i\hat{a}^{\dagger}) |0\rangle$$

$$= \frac{1}{\sqrt{2}} (\hat{a}^{\dagger} \hat{b}^{\dagger} - i\hat{a}^{\dagger 2} - i\hat{b}^{\dagger 2} + (-i)^2 \hat{b}^{\dagger} \hat{a}^{\dagger}) |0\rangle$$

Overall phase \rightarrow

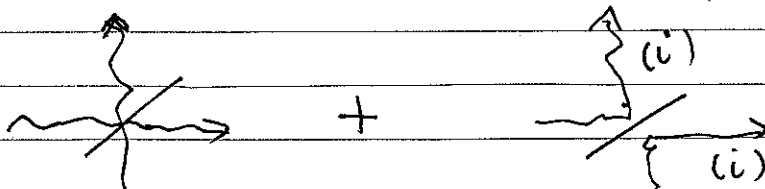
$$= \frac{-i}{\sqrt{2}} (\hat{a}^{\dagger 2} + \hat{b}^{\dagger 2}) |0\rangle + \frac{1}{\sqrt{2}} (\hat{a}^{\dagger} \hat{b}^{\dagger} - \hat{b}^{\dagger} \hat{a}^{\dagger}) |0\rangle$$

$$\Rightarrow |2\rangle^{(out)} = \frac{1}{\sqrt{2}} (|2\rangle_a \otimes |0\rangle_b + |0\rangle_a \otimes |2\rangle_b)$$

Since $[\hat{a}^{\dagger}, \hat{b}^{\dagger}] = 0$

Thus both photons go off together.

We see that there is destructive interference for the two processes below



both transmitted

both reflected. Each picks up $\frac{\pi}{2}$ phase shift, causing destructive interference.

Problem 2: Collapse and revival in the Jaynes-Cummings model

We consider a two-level atom coupled to a single mode of a high-Q cavity on resonance. The field is initially prepared in a coherent state and the atom in the ground state: $|\Psi(0)\rangle = |g\rangle \otimes |\alpha\rangle$. The joint atom-field state then evolves according to the Jaynes-Cummings Hamiltonian (neglecting here any spontaneous emission into other modes, or cavity losses).

(a) To evolve the state, we can first decompose the initial state into the eigenstates of J.C. Hamiltonian: $\{|+\rangle, |-\rangle\}$ with eigenvalues: $E_{\pm(n)} = n\hbar\omega_c \pm \hbar\sqrt{n}g$ ← coupling constant

Within the 2D-subspace, the time evolution operator is

$$\hat{U}_n = e^{-iE_{+(n)}t/\hbar} |+\rangle\langle +| + e^{-iE_{-(n)}t/\hbar} |-\rangle\langle -| \\ = e^{-in\omega_c t} \left[e^{-i\sqrt{n}gt} |+\rangle\langle +| + e^{+i\sqrt{n}gt} |-\rangle\langle -| \right] \quad n > 0$$

$$\Rightarrow \hat{U}_n |g\rangle \otimes |n\rangle = e^{-in\omega_c t} \left[\cos(\sqrt{n}gt) |g\rangle \otimes |n\rangle + i \sin(\sqrt{n}gt) |e\rangle \otimes |n-1\rangle \right]$$

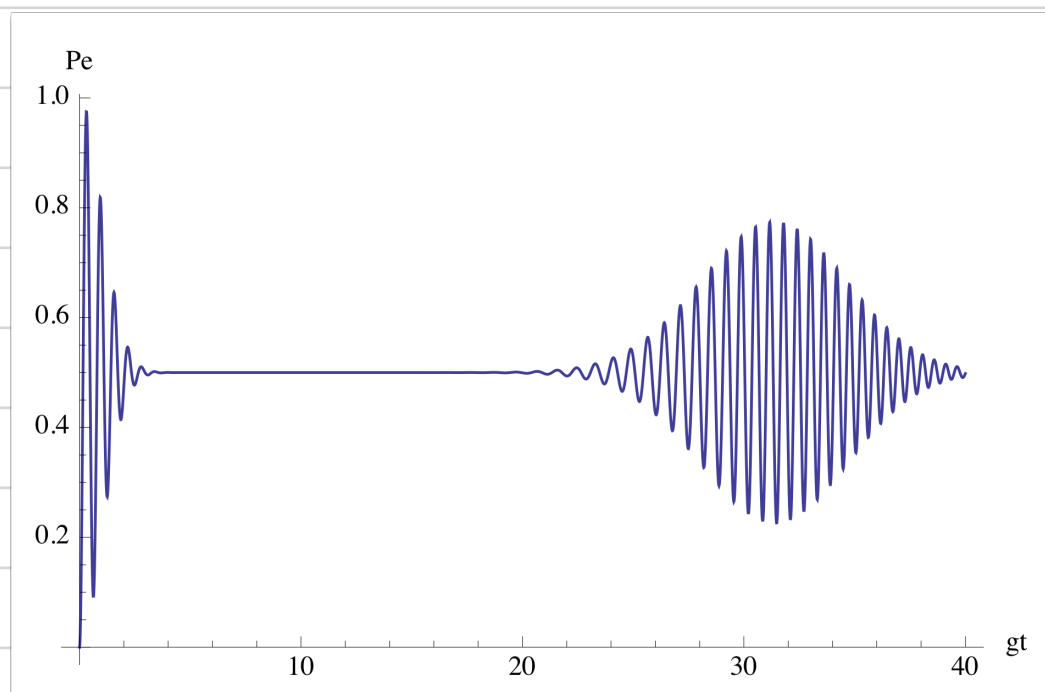
$$|\Psi(0)\rangle = |g\rangle \otimes |\alpha\rangle = |g\rangle \otimes \sum_n C_n |n\rangle = \sum_{n=0}^{\infty} C_n |g\rangle \otimes |n\rangle \quad \text{where } C_n = \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2} \\ = C_0 |g\rangle \otimes |0\rangle + \sum_{n=1}^{\infty} C_n |g\rangle \otimes |n\rangle$$

$$\Rightarrow |\Psi(t)\rangle = \bigoplus_n \hat{U}_n |\Psi(0)\rangle = \\ = C_0 |g\rangle \otimes |0\rangle + \sum_{n=1}^{\infty} C_n e^{-in\omega_c t} \left[\cos(\sqrt{n}gt) |g\rangle \otimes |n\rangle + i \sin(\sqrt{n}gt) |e\rangle \otimes |n-1\rangle \right]$$

The probability to find the atom in the excited state, irrespective of photon number n is $P_e(t) = \sum_{n=0}^{\infty} |\langle e, n | \Psi(t) \rangle|^2$ (sum over n)

$$\Rightarrow P_e(t) = \sum_{n=1}^{\infty} |C_n|^2 \sin^2(\sqrt{n}gt) = \sum_{n=1}^{\infty} \frac{\bar{n}^n}{n!} e^{-\bar{n}} \sin^2(\sqrt{n}gt)$$

(b) Shown is a numerical plot of $P_e(t)$ for $\bar{n}=25$ as a function of $0 \leq gt \leq 40$.

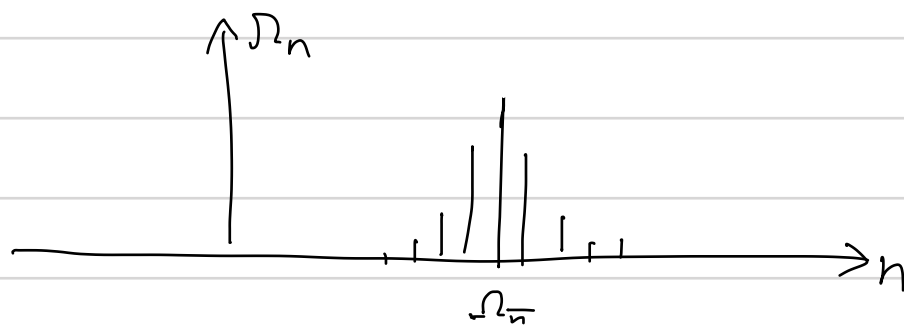


This is the famous Jaynes-Cummings "collapse and revival." The collapse is expected due to a spread in the Rabi-frequencies $\Omega_n = \sqrt{n} 2g$, ($\Delta n^2 = \bar{n}$, Poisson number fluctuations). Given a spread in photon number, $\Delta n = \sqrt{\bar{n}}$, the spread in Rabi frequencies is

$$\Delta \Omega \sim |\Omega_{\bar{n}+\sqrt{\bar{n}}} - \Omega_{\bar{n}-\sqrt{\bar{n}}}| = 2g\sqrt{\bar{n}+\sqrt{\bar{n}}} - 2g\sqrt{\bar{n}-\sqrt{\bar{n}}} \approx 2g\sqrt{\bar{n}} \left(1 + \frac{1}{2\sqrt{\bar{n}}}\right) - 2g\sqrt{\bar{n}} \left(1 - \frac{1}{2\sqrt{\bar{n}}}\right) \\ \approx 2g \quad \text{for } \bar{n} \gg 1$$

\Rightarrow Collapse time $t_c \sim \frac{1}{\Delta \Omega} \sim \frac{1}{2g}$ (as seen in the plot)

(c) The revival, by contrast, is a much more subtle effect that is a unique signature of quantum fluctuations of the cavity field. That is the Rabi frequencies are associated with a discrete set photons, rather than the continuum of intensities. Thus the probability $P_e(t)$ is a Fourier sum rather than an integral.



At times such that $(\Omega_n - \Omega_{n+1})t_m \cong 2\pi m$ we expect a "rephasing" and thus a revival of the oscillations.

$$\Rightarrow t_m \cong \frac{2\pi m}{2g\sqrt{n} - 2g\sqrt{n+1}} \cong \frac{\pi m}{g\sqrt{n} (1 - (1 - \frac{1}{2n}))} = \frac{2\pi m \sqrt{n}}{g}$$

\Rightarrow First revival: $t_1 \cong \frac{2\pi\sqrt{n}}{g} \Rightarrow gt_1 \cong 2\pi\sqrt{n} \cong 31.4$, as seen in the plot.

Note: The revival is not perfect because the sum is infinite. A finite Fourier series will always have perfect revivals.